Relativistic vs. Newtonian Kinetic Energy

Abstract

The value of the kinetic energy of a moving body of momentum P is not the same value in Special Relativity (SRT) as it is in Newtonian Mechanics (NM). This paper examines, explains, and corrects this dilemma using two methods.

Kinetic Energy in SRT and NM

Kinetic energy of a freely moving body, which will be denoted by K_S in SRT and K_N in NM, is the product of the force F applied times the distance x' over which it is applied, or

$$K = F x' \qquad 1$$

Net applied force results in change in momentum P.

$$F = dP/dt \qquad 2$$

So

$$K_S = (dP/dt)x' \qquad 3$$

in SRT. Also, NM kinetic energy must equal

$$K_N = (dP/dt)x' \qquad 4$$

In SRT

$$I^2 = (ct)^2 - x^2 = (ct')^2 - x'^2 = \ldots \qquad 5$$

where I is the Relativistic Interval Equation which is the same in all reference frames for any two events, c is the speed of light, and t in these examples is the time during which the light originating in the non-moving reference frame at x=0 has traveled the distance I at speed c in that frame. The x' is the distance an object or reference frame has moved perpendicularly while the light has moved the distance I. The x=0 when considering the light rays moving perpendicular to the direction of motion of x'. Then the formula looks like

$$I^2 = (ct)^2 = (ct')^2 - x'^2 = (ct'')^2 - x''^2 \ldots \qquad 6$$

The x (not x') in these examples will not change because it is the non-moving reference frame, x=0 .

Speed is distance per unit time. If you know the distance an object traveled, x', the distance a timing signal (light) traveled, I, and the speed, c, of that timing signal, then NM velocity, V, would be

$$V = (x'/I)c = (x'/ct)c = x'/t \qquad 7$$

Momentum P in NM is

$$P = mV = m(x'/t) = m(x'/I)c \qquad 8$$

According to SRT

$$x' = \gamma(x+vt) \qquad 9$$

$$t' = \gamma(t + (v/c^2)x) \qquad 10$$

so for x=0 , in which case I=ct ,

$$x' = \gamma vt = \gamma(v/c)ct = \gamma(v/c)I \qquad 11$$

$$t' = \gamma t \qquad 12$$

where x' is the distance the moving frame traveled while the light traveled the distance I at speed c, γ is the gamma function, $\gamma = 1/(1-(v/c)^2)^{1/2}$, t' is calculated from (I/c) according to the SRT Eq. 12, and v is the SRT velocity (not the same as NM velocity V), which is x'/t' by dividing Eq. 11 by Eq. 12 or

$$x'/t' = \gamma vt/\gamma t = v \qquad 13$$

By Eq. 11

$$x'/I = \gamma(v/c) \qquad 14$$

and momentum P in SRT has the same value as in NM, namely

$$P = m(x'/I)c = m\gamma(v/c)c = m\gamma v \qquad 15$$

using Eqs. 8 and 14. Since in NM $P=mV$, then

$$V = \gamma v \qquad 16$$

$$x'/I = (\gamma v)/c = \gamma(v/c) = (P/m)/c \qquad 17$$

meaning the ratio of the distance the object traveled to the distance the light traveled is $(P/m)/c$. Therefore, distance traveled x' would be

$$x' = (P/m)I/c = (1/m)Pct/c = (1/m)Pt \quad . \qquad 18$$

However, P is varying at the rate $(dP/dt) = F =$ a constant. P at any given time is

$$P = (dP/dt)t \qquad 19$$

Using Eqs. 18 and 19

$$dx' = (1/m)(dP/dt)t\ dt \qquad 20$$

$$x' = (1/m)(dP/dt) \int t\ dt \qquad 21$$

$$x' = (1/2m)(dP/dt)\ t^2 \qquad 22$$

and since $K=Fx'$, from Eqs. 3 and 22 one gets

$$K_S = (dP/dt)\ x' = (1/2m)\ (dP/dt)^2\ t^2 \qquad 23$$

which is

$$K_S = (1/2m)\ (m\ d(\gamma v)/dt)^2\ t^2 = \tfrac{1}{2}\ m(\gamma v)^2 \qquad 24$$

in SRT. In NM it is

$$K_N = (1/2m)\ (m\ dV/dt)^2\ t^2 = \tfrac{1}{2}\ mV^2 \qquad 25$$

Another Method

The usual formula given for K_S in SRT is

$$K_S = mc^2\ (\gamma - 1) \quad . \qquad 26$$

This is derived from

$$dK = F\ dx' = (dP/dt)\ dx' = (dx'/dt)\ dP \quad . \qquad 27$$

The correct $P=m\gamma v$ was used for the P of dP. However, the (dx'/dt) which equals P/m, the correct form and the same as NM velocity, is not what was used for the integration. Instead, $P/\gamma m$ was used for (dx'/dt) which is v or (dx'/dt'), where the denominator is the dilated time, was used. Thus the integration result of $mc^2(\gamma-1)$ for K_S.

If calculated correctly by this "$(dx'/dt)\ dP$" method of Eq. 27, it looks like this.

$$dK_S = F\ dx' = (dP/dt)\ dx' = (dx'/dt)\ dP = (1/m)\ P\ dP \qquad 28$$

$$K_S = (1/m) \int P dP \qquad 29$$

Therefore, by the method used to calculate the "$(\gamma\text{-}1)$" version (Eq. 27) for K_S, when done properly using $(1/m)P$ instead of $(1/m)(P/\gamma)=v$, the result is

$$K_S = (1/2m)\ P^2 = ½\ m(\gamma v)^2 = ½\ mc^2(\gamma(v/c))^2 \qquad 30$$

just as found above in Eq 24. Which, by the way, since $P=mV$ in NM, is

$$K_N = (1/2m)\ P^2 = ½\ mV^2 = ½\ mc^2(V/c)^2 = ½\ mc^2(\gamma(v/c))^2 \qquad 31$$

Conclusion

The correct equation for kinetic energy in Special Relativity (SRT) is

$$K_S = ½\ m(\gamma v)^2 = ½\ mc^2(\gamma(v/c))^2 = ½\ mc^2(V/c)^2$$

equal to the Newtonian Mechanics (NM) value

$$K_N = ½\ mV^2 = ½\ mc^2(V/c)^2$$

The difference between the old $K_S=mc^2(\gamma-1)$ and the correct $K_S= ½\ m(\gamma v)^2 = ½\ mc^2(\gamma(v/c))^2$ is $< 1\%$ for $(v/c) < 0.2$.